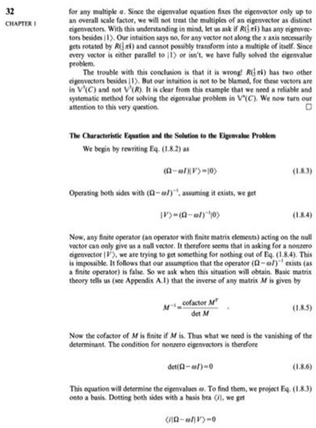

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CHAPTER 1

for any multiple a. Since the eigenvalue equation fixes the eigenvector only up to an overall scale factor, we will not treat the multiples of an eigenvector as distinct eigenvectors. With this understanding in mind, let us ask if $R(\frac{1}{2}\pi\mathbf{i})$ has any eigenvectors besides $|1\rangle$. Our intuition says no, for any vector not along the x axis necessarily gets rotated by $R(\frac{1}{2}\pi\mathbf{i})$ and cannot possibly transform into a multiple of itself. Since every vector is either parallel to $|1\rangle$ or isn't, we have fully solved the eigenvalue problem.

The trouble with this conclusion is that it is wrong! $R(\frac{1}{2}\pi\mathbf{i})$ has two other eigenvectors besides $|1\rangle$. But our intuition is not to be blamed, for these vectors are in $\mathbb{V}^3(C)$ and not $\mathbb{V}^3(R)$. It is clear from this example that we need a reliable and systematic method for solving the eigenvalue problem in $\mathbb{V}^n(C)$. We now turn our attention to this very question. □

The Characteristic Equation and the Solution to the Eigenvalue Problem

We begin by rewriting Eq. (1.8.2) as

$$(\Omega-\omega I)|V\rangle=|0\rangle \quad (1.8.3)$$

Operating both sides with $(\Omega-\omega I)^{-1}$, assuming it exists, we get

$$|V\rangle=(\Omega-\omega I)^{-1}|0\rangle \quad (1.8.4)$$

Now, any finite operator (an operator with finite matrix elements) acting on the null vector can only give us a null vector. It therefore seems that in asking for a nonzero eigenvector $|V\rangle$, we are trying to get something for nothing out of Eq. (1.8.4). This is impossible. It follows that our assumption that the operator $(\Omega-\omega I)^{-1}$ exists (as a finite operator) is false. So we ask when this situation will obtain. Basic matrix theory tells us (see Appendix A.1) that the inverse of any matrix M is given by

$$M^{-1}=\frac{\text{cofactor } M^T}{\det M} \quad (1.8.5)$$

Now the cofactor of M is finite if M is. Thus what we need is the vanishing of the determinant. The condition for nonzero eigenvectors is therefore

$$\det(\Omega-\omega I)=0 \quad (1.8.6)$$

This equation will determine the eigenvalues ω. To find them, we project Eq. (1.8.3) onto a basis. Dotting both sides with a basis bra $\langle i|$, we get

$$\langle i|\Omega-\omega I|V\rangle=0$$